Eric Gallimore
SIO221A Final Exam

Task 1

To determine an expression for $\widehat{H}(\sigma)$, we begin with a cross-spectral estimate, the general form for which is

$$C_{Xy} = \frac{a_X^* a_y}{\Delta\sigma}$$

Now, we recognize that, in our measurements, the Fourier coefficients a_X are actually composed of two components: the signal, a_x, and some incoherent noise, a_n. When substituted, this yields

$$C_{(x+n)y} = \frac{(a_x + a_n)^* a_y}{\Delta\sigma}$$

Or, with algebraic rearrangement,

$$C_{(x+n)y} = \frac{a_x^* a_y + a_n^* a_y}{\Delta\sigma}$$

Now, we want to determine the frequency response. We begin with the general form:

$$H(\sigma) = \frac{C_{Xy}(\sigma)}{E_X(\sigma)}$$

In addition to incorporating the above expression $C_{(x+n)y}$ as the cross-spectral estimate, we recognize that the variance spectrum is actually composed of the superposition of two components: the spectrum of the signal, E_x, and the spectrum of the noise, E_n. Therefore,

$$\widehat{H}(\sigma) = \frac{C_{(x+n)y}}{E_x + E_n}$$

This can be rewritten to develop a noise-to-signal ratio term, E_n/E_x:

$$\widehat{H}(\sigma) = \frac{C_{(x+n)y}}{E_x(1 + \frac{E_n}{E_x})}$$

At this point, it is useful to note (as shown, expanded, above) that

$$C_{(x+n)y} = C_{xy} + C_{ny}$$

Since we believe that the noise is uncorrelated with our signal (we have no reason to think otherwise), by the definition of the cross spectrum,

$$C_{ny} = 0$$

And, therefore

$$C_{(x+n)y} = C_{xy}$$

So,

$$\hat{H}(\sigma) = \frac{C_{xy}}{E_x(1 + \frac{E_n}{E_x})}$$

Now, we determine the relationship between $\hat{H}(\sigma)$ and $H(\sigma)$.

$$\frac{\hat{H}(\sigma)}{H(\sigma)} = \frac{E_x}{C_{xy}} \frac{C_{xy}}{E_x(1 + \frac{E_n}{E_x})}$$

$$\frac{\hat{H}(\sigma)}{H(\sigma)} = \frac{1}{(1 + \frac{E_n}{E_x})}$$

$$\hat{H}(\sigma) = \frac{H(\sigma)}{(1 + \frac{E_n}{E_x})} = \frac{1}{1 + \frac{E_n}{E_x}} H(\sigma)$$

From this, we can see that the noise decreased the gain of $\hat{H}(\sigma)$ relative to $H(\sigma)$ by a factor of (1+NSR). It didn't change the phase.

(The effect of other inputs of noise in this system is discussed in the course notes and is not relevant at this point since the noise is incoherent.)